

Classical and SUSY solutions of the Boiti-Leon-Manna-Pempinelli equation.

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Abstract. In this paper, we propose the study of the Boiti-Leon-Manna-Pempinelli equation from two point of views: the classical and supersymmetric cases. In the classical case, we construct new solutions of this equation from Wronskian formalism and Hirota method. We, then, introduce a $\mathcal{N} = 1$ supersymmetric extension of the Boiti-Leon-Manna-Pempinelli equation. We thus produce a bilinear form and give multisolitons and superpartner solutions. As an application, we produce Bäcklund transformations.

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1. Introduction

Soliton equations has been largely studied in the past in terms of integrability conditions and solutions [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. These solutions have been discovered through numerous techniques such as Bäcklund transformations, inverse scattering methods, Lax pairs formulation, G'/G -expansions, Wronskian formulation and bilinear Hirota method. The later ones will be of interest in this paper. To this instance, the Hirota derivative is defined as

$$\mathcal{D}_x^n(f \cdot g) = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)|_{x=x_1=x_2}. \quad (1)$$

In this paper, we consider the Boiti-Leon-Manna-Pempinelli (BLMP) equation

$$u_{yt} + u_{xxxy} - 3u_x u_{xy} - 3u_y u_{xx} = 0, \quad (2)$$

where $u = u(x, y, t)$ and subscripts represent partial differentiation with respect to the given variable. This equation has received small attention [11] and here we propose to give some new classical solutions through the Hirota and Wronskian methods. To do so, we note that the BLMP equation is strongly related to the Korteweg-de Vries (KdV) equation. Indeed, let us take

$$u(x, y, t) = p(z, t) = p(x + q(y), t), \quad (3)$$

where q is an arbitrary function of the variable y , then p satisfies

$$p_{zt} + p_{zzzz} - 6p_z p_{zz} = 0.$$

Defining $h = p_z$, we see that h satisfies the KdV equation, which has the following bilinear form [4, 8, 9]

$$\mathcal{D}_z(\mathcal{D}_t + \mathcal{D}_z^3)(f \cdot f) = 0, \quad (4)$$

using the change of variable $h = -2\partial_z^2 \log f$. Thus $p = -2\partial_z \log f$ satisfies the BLMP equation for $f(z, t)$ a solution of the above bilinear equation associated to the KdV equation. This suggests that the BLMP equation possesses multisoliton and rational similarity solutions and many other solutions as we will demonstrate.

Without doing the reduction (3) in $u(x, y, t)$ but using the change of variable given above, we can bilinearize, in general, the BLMP equation (2) by casting

$$u(x, y, t) = -2\partial_x \log f(x, y, t) - m(y), \quad (5)$$

where $m = m(y)$ is an arbitrary function of y . One then gets the bilinear equation

$$(\mathcal{D}_y(\mathcal{D}_t + \mathcal{D}_x^3) + 3m'(y)\mathcal{D}_x^2)(f \cdot f) = 0. \quad (6)$$

After considering the classical analysis of the BLMP equation, it is natural to ask what would be its supersymmetric (SUSY) counterpart. SUSY extensions of nonlinear partial differential equation (PDE) has, in the past few decades, been largely studied in terms of integrability and solutions, see for example [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 27, 28, 29, 30]. Here, we consider for the first time a $\mathcal{N} = 1$ SUSY extension of the BLMP equation (2).

To produce a $\mathcal{N} = 1$ SUSY extension of the BLMP equation, we extend, as usual, the space (x, y, t) to the superspace $(x, y, t; \theta)$ [25] where θ is a Grassmann variable which satisfies $\theta^2 = 0$. As usual, we consider a fermionic superfield $\Phi(x, y, t; \theta)$ to get non trivial SUSY extensions [12, 13, 15, 16, 18]. A $\mathcal{N} = 1$ SUSY extension of the BLMP equation (2) is given as

$$D_y^2 \Phi_t + D_y^2 D_x^6 \Phi - 3D_x D_y (D_x^3 \Phi D_y D_x \Phi) = 0, \quad (7)$$

where D_x and D_y denotes the supercovariant derivatives defined as

$$D_x = \partial_\theta + \theta \partial_x, \quad D_y = \partial_\theta + \theta \partial_y, \quad D_x^2 = \partial_x, \quad D_y^2 = \partial_y. \quad (8)$$

Using the finite Taylor expansion

$$\Phi(x, y, t; \theta) = \xi(x, y, t) + \theta u(x, y, t), \quad (9)$$

where ξ is a complex-valued fermionic (odd) function and u is a complex-valued bosonic (even) function, we get a set of two equations

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} + 3\xi_x\xi_{xxx} = 0, \quad (10)$$

$$\xi_{yt} + \xi_{xxxy} - 3u_{xy}\xi_x - 3u_x\xi_{xy} = 0. \quad (11)$$

In the fermionic limit where ξ is set to zero, we retrieve the classical BLMP equation as expected.

As in the classical case, the SUSY BLMP equation is related to the SUSY KdV equation. Indeed, supposing that $\xi(x, y, t) = \sqrt{q'(y)}\chi(z, t)$ and $u(x, y, t) = v(z, t)$ where $z = x + q(y)$, then the above set of equations leads to

$$v_{zt} + v_{zzzz} - 6v_zv_{zz} + 3\chi_z\chi_{zzz} = 0, \quad (12)$$

$$\chi_t + \chi_{zzz} - 3\chi_zv_z = 0. \quad (13)$$

In the fermionic limit, we get the classical KdV equation for the variable v_z . The equations (12) and (13) becomes the SUSY KdV equation first proposed by Manin and Radul [31] after identifying $\chi_z = \varphi$. We thus get a series of solutions of the SUSY BLMP equation using this procedure.

In section 2, we give new classical solutions of the BLMP equation using the Wronskian and Hirota bilinear formulation. In section 3, we introduce the SUSY Bell polynomials which will be the key ingredient of our analysis. In section 4, we give solutions of the SUSY BLMP equation, *e.g.* soliton and periodic solutions. There is also an attempt to find the superpartner solutions of the given classical solutions u of the BLMP equation. In section 5, we produce bilinear Bäcklund transformations using the SUSY Bell polynomials described in section 3.

2. New classical solutions

In the introduction, we have shown a link between the BLMP equation and the KdV equation. It means that, for example, we can produce rational similarity solutions for

the BLMP equation based on the corresponding ones for KdV. Indeed, for the rational similarity solutions [1, 2, 4, 5, 7, 10], we get

$$u_n(x, y, t) = \phi_n(z, t) = -2 \frac{d}{dz} \log \mathcal{Q}_n(z(3t)^{-\frac{1}{3}}),$$

where the functions \mathcal{Q}_n are known as the Yablonskii-Vorob'ev polynomials and are defined recursively [7] as

$$\mathcal{Q}_{n+1} \mathcal{Q}_{n-1} = z \mathcal{Q}_n^2 - 4(\mathcal{Q}_n \mathcal{Q}_{n,zz} - \mathcal{Q}_{n,z}^2),$$

with $\mathcal{Q}_0(z) = 1$ and $\mathcal{Q}_1(z) = z$. For example, we get the solutions

$$u_2(z, t) = -\frac{6z^2}{12t + z^3}, \quad u_3(z, t) = \frac{12(30tz^2 + z^5)}{720t^2 - 60tz^3 - z^6}.$$

This infinite series of rational similarity solutions have not been obtained before.

We can exhibit soliton solutions. Indeed, for the N -soliton solution one gets for the function f [1, 8]

$$f_N = \sum_{\mu=0,1} \exp \left(\sum_{i=1}^N \mu_i \psi_i + \sum_{i<j} a_{ij} \mu_i \mu_j \right), \quad (14)$$

where $\psi_i = \kappa_i z - \kappa_i^3 t$ and $e^{a_{ij}} = A_{ij} = \left(\frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j} \right)^2$. The above solutions has been deduced from the reduction of the BLMP equation to the KdV equation via (3).

Using a Wronskian formulation for the KdV equation [3, 6, 8], we get rational, positon, negaton and complexiton solutions of the BLMP equation. To this instance, we look for solutions $f = f(z, t)$ of the following form

$$f = \det(H^{(0)}, H^{(1)}, \dots, H^{(N-1)}), \quad (15)$$

where $H = H^{(0)} = (h_1, h_2, \dots, h_N)^T$ and $H^{(i)} = \frac{\partial^i}{\partial z^i} H$ for $i = 1, 2, \dots, N-1$. The resulting function $u = -2\partial_z \log f$ is called a Wronskian solution of order N . The Wronskian solution requires [3, 6, 8, 11] that the functions h_i satisfy the following linear PDE's

$$-h_{i,zz} = \lambda_i h_i, \quad h_{i,t} = -4h_{i,zzz}, \quad (16)$$

for $i = 1, 2, \dots, N$. When λ_i is real, we get three cases of solutions: rational ($\lambda_i = 0$), positon ($\lambda_i > 0$) and negaton ($\lambda_i < 0$). For the complexiton solution, the eigenvalues λ_i are supposed to be complex.

Here we give some examples and refer the interested readers to [3, 6, 8, 11] for further discussions on the Wronskian solutions.

A negaton solution of second order

$$u_n(z, t, \gamma_1) = \frac{8\gamma_1 \cosh^2(\gamma_1(z - 4\gamma_1^2 t))}{2\gamma_1(12\gamma_1^2 t - z) - \sinh(2\gamma_1(z - 4\gamma_1^2 t))}, \quad (17)$$

a positon of second order

$$u_p(z, t, \gamma_1) = -\frac{8\gamma_1 \cos^2(\gamma_1(z + 4\gamma_1^2 t))}{2\gamma_1(12\gamma_1^2 t + z) + \sin(2\gamma_1(z + 4\gamma_1^2 t))} \quad (18)$$

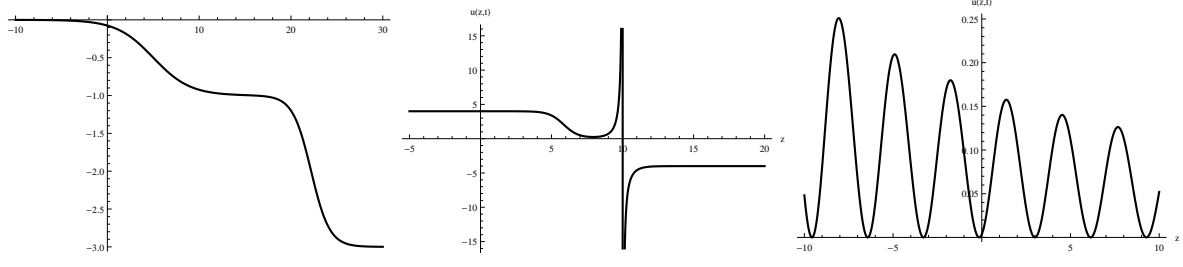


Figure 1. From left to right, a 2-soliton solution (14) for $N = 2$ ($\kappa_1 = \frac{1}{2}$, $\kappa_2 = 1$) and $t = 20$, a negaton of second order (17) ($\gamma_1 = 1$) and a positon of second order (18) ($\gamma_1 = 1$) for $z = x + q(y)$ and $t = 2$.

and a complexiton

$$u_c(z, t, \eta) = \frac{4i\text{Im}(\eta) \cosh(\bar{\chi}) \cosh(\chi)}{\sqrt{\eta} \cosh(\bar{\chi}) \sinh(\chi) - \sqrt{\bar{\eta}} \cosh(\chi) \sinh(\bar{\chi})}, \quad (19)$$

where $\chi = \sqrt{\eta}(z - 4\eta t)$, $\bar{\chi} = \sqrt{\bar{\eta}}(z - 4\bar{\eta}t)$ and $\eta = \alpha + i\beta$. There is numerous more solutions to the BLMP equation such as interacting solutions. For example, one can construct a rational-soliton (u_{rs}) and rational-positon (u_{rp}) solutions using Wronskian formulation [3, 6, 8, 11]. One gets

$$u_{rs}(z, t, \gamma_1) = \frac{-2\gamma_1^2 z}{\gamma_1 z \tanh(\gamma_1(z - 4\gamma_1^2 t)) - 1} \quad (20)$$

and

$$u_{rp}(z, t, \gamma_1) = \frac{-2\gamma_1^2 z}{\gamma_1 z \tan(\gamma_1(z + 4\gamma_1^2 t)) + 1} \quad (21)$$

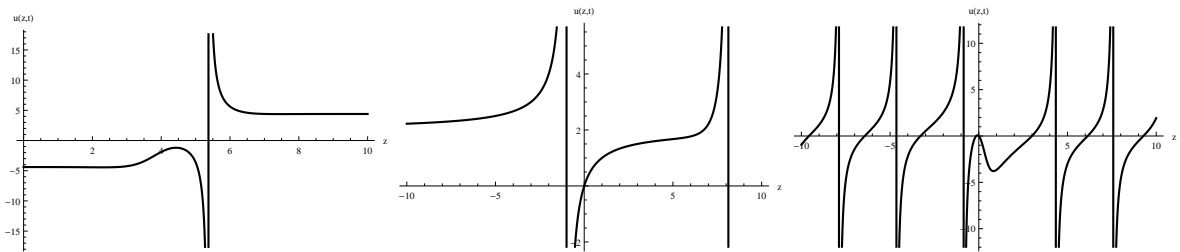


Figure 2. From left to right, a complexiton (19) ($\alpha = \beta = 1$), a rational-soliton (20) and a rational-positon (21) solutions for $z = x + q(y)$, $t = 2$ and $\gamma_1 = 1$.

In the general case, one could be interested in solving the bilinear equation (6) and retrieve, for example, a general travelling wave solution. Indeed, if the following constraints are supposed

$$f = f(w) = f\left(ax + \frac{1}{a}m(y) - 4a^3t\right), \quad f_{ww} = \alpha f_w + \frac{1}{4}(1 - \alpha^2)f,$$

where α is an arbitrary complex constant, then the bilinear form (6) is automatically satisfied and we get a solution of the BLMP equation

$$u = -2a\partial_w \log\left(c_1 e^{\frac{1}{2}(1+\alpha)w} + c_2 e^{\frac{1}{2}(-1+\alpha)w}\right) - m(y), \quad (22)$$

where c_1 and c_2 are integration constants and $m = m(y)$ is an arbitrary function of the variable y used in (5) to bilinearize the BLMP equation. At our knowledge, the above solution is new. Here we give an example for $\alpha = 0$ and $c_1 = c_2 = 1$, one gets the kink-type solution

$$u(x, y, t) = -a \tanh\left(\frac{w}{2}\right) - m(y).$$

Most of the above solutions are new and will be useful in subsequent sections to construct superpartners of the supersymmetric BLMP equation.

3. SUSY extension of the Bell polynomials

The Bell polynomials are known as an alternative to tedious calculations of finding a bilinear form of an equation [11, 22]. Indeed, we are able to rewrite the equation in terms of the Bell polynomials and to get simultaneously the change of variable and the bilinear form of the studied equation. For more details, we refer the reader to [22].

For our present analysis, we introduce a particular case of the SUSY Bell polynomials. We refer the interested reader to [22] for a more general introduction to the SUSY Bell polynomials.

Let $A(X; \theta)$ be a superdifferentiable bosonic superfield [25] where $X = (x, y, t)$. The super Bell polynomials are defined as

$$Y_{l \cdot X, (k_x, k_y)}(A) = Y_{l \cdot X, (k_x, k_y)}(A_{l \cdot X, (k_x, k_y)}) = e^{-A} D_x^{k_x} D_y^{k_y} \partial_x^{l_x} \partial_y^{l_y} \partial_t^{l_t} e^A, \quad (23)$$

where $l \cdot X = (l_x x, l_y y, l_t t)$ and D_x and D_y are the supercovariant derivatives defined in (8).

The super Bell polynomials enjoys a remarkable property given by

$$Y_{l \cdot X, (k_x, k_y)}(A)|_{A=\ln f} = \frac{f_{l \cdot X, (k_x, k_y)}}{f} := \frac{D_x^{k_x} D_y^{k_y} \partial_x^{l_x} \partial_y^{l_y} \partial_t^{l_t} f(X; \theta)}{f(X; \theta)}. \quad (24)$$

The super binary polynomials are defined, using the super Bell polynomials, as

$$\mathcal{Y}_{l \cdot X, (k_x, k_y)}(v, w) = Y_{l \cdot X, (k_x, k_y)}(A_{s \cdot X, (n_x, n_y)}), \quad (25)$$

where the different derivatives of A are replaced by the superfields v and w using the following procedure

$$A_{s \cdot X, (n_x, n_y)} = \begin{cases} v_{s \cdot X, (n_x, n_y)}, & s_x + s_y + s_t + n_x + n_y \text{ is odd} \\ w_{s \cdot X, (n_x, n_y)}, & s_x + s_y + s_t + n_x + n_y \text{ is even} \end{cases}$$

As an example, we have that

$$\begin{aligned} Y_{(3x, 0, 0), (0, 0)}(A) &= A_{xxx} + 3A_x A_{xx} + A_x^3, \\ \mathcal{Y}_{(3x, 0, 0), (0, 0)}(v, w) &= v_{xxx} + 3v_x w_{xx} + v_x^3. \end{aligned}$$

The super binary Bell polynomials are related to the SUSY Hirota derivative [22]. Indeed, we have the following relation

$$\mathcal{Y}_{l \cdot X, (k_x, k_y)}\left(v = \ln \frac{f}{g}, w = \ln fg\right) = (fg)^{-1} \mathcal{S}_x^{k_x} \mathcal{S}_y^{k_y} \mathcal{D}_x^{l_x} \mathcal{D}_y^{l_y} \mathcal{D}_t^{l_t} (f \cdot g), \quad (26)$$

where the super Hirota derivative is defined as

$$\mathcal{S}_\mu(f \cdot g) = (D_\mu - D_{\mu'})f(X, \theta)g(X', \theta')|_{X=X', \theta'=\theta}. \quad (27)$$

The super P -polynomials [22] are a particular class of (26) where we take $f = g$. We have that

$$P_{l \cdot X, (k_x, k_y)}(w) = \mathcal{Y}_{l \cdot X, (k_x, k_y)}(0, w = 2 \ln g) = g^{-2} \mathcal{S}_x^{k_x} \mathcal{S}_y^{k_y} \mathcal{D}_x^{l_x} \mathcal{D}_y^{l_y} \mathcal{D}_t^{l_t} (g \cdot g). \quad (28)$$

A direct consequence of (26) is that $P_{l \cdot X, (k_x, k_y)}(w) = 0$ whenever $l_x + l_y + l_t + k_x + k_y$ is odd.

4. Bilinear form, multisolitons and superpartners

In this section, we propose a bilinear form of the SUSY BLMP equation (7) and give multisolitons and superpartner solutions. In order to get this form, let us cast the change of variable $\Phi = D_x F + \Lambda(y; \theta)$ where F is a bosonic superfield, Λ is a fermionic superfield and integrate with respect to D_x and secondly by D_y . We thus get the fermionic equation

$$D_y(F_t + D_x^6 F) - 3(D_x^4 F)(D_y D_x^2 F) - 3(D_y D_x \Lambda)F_{xx} = 0, \quad (29)$$

where the constants of integration as been set to zero and we recall that $D_x^2 = \partial_x$. Using the following P-polynomials

$$P_{(0,0,t),(0,1)}(w) = D_y w_t, \quad P_{(3x,0,0),(0,1)}(w) = D_y w_{xxx} + 3w_{xx} D_y w_x,$$

equation (29) can be rewritten as the polynomial equation

$$P_{(0,0,t),(0,1)}(-F) + P_{(3x,0,0),(0,1)}(-F) - 3(D_y D_x \Lambda)P_{(2x,0,0),(0,0)}(-F) = 0.$$

We thus cast $F = -2 \ln g$ and we get the bilinear form of the SUSY BLMP equation (7) given by

$$(\mathcal{S}_y \mathcal{D}_t + \mathcal{S}_y \mathcal{D}_x^3 - 3(D_y D_x \Lambda) \mathcal{D}_x^2)(g \cdot g) = 0. \quad (30)$$

Note that if $\Lambda(y; \theta) = \eta(y) + \theta c(y)$ then we get $D_y D_x \Lambda = \theta c'(y)$ and, in terms of components, if $g = g^b + \theta g^f$, the bilinear equation (30) becomes

$$\begin{aligned} \mathcal{Q}_1(\mathcal{D}_x, \mathcal{D}_y, \mathcal{D}_t)(g^b \cdot g^b) &= (\mathcal{D}_y(\mathcal{D}_t + \mathcal{D}_x^3) - 3c'(y) \mathcal{D}_x^2)(g^b \cdot g^b) = 0, \\ \mathcal{Q}_2(\mathcal{D}_x, \mathcal{D}_y, \mathcal{D}_t)(g^b \cdot g^f) &= (\mathcal{D}_t + \mathcal{D}_x^3)(g^b \cdot g^f) = 0. \end{aligned}$$

In order to find the dispersion relations of g^b and g^f , one as to solve the set of equations

$$\mathcal{Q}_1(\kappa, \rho, \omega) = 0, \quad \mathcal{Q}_2(\kappa, \rho, \omega) = 0,$$

which is equivalent to

$$\rho(\omega + \kappa^3) - 3c'(y)\kappa^2 = 0, \quad \omega + \kappa^3 = 0.$$

Thus we get $c'(y) = 0$. This shows that to get non-trivial multisoliton solutions, one as to consider the super bilinear form

$$\mathcal{S}_y(\mathcal{D}_t + \mathcal{D}_x^3)(g \cdot g) = 0. \quad (31)$$

4.1. Multisolitons

As usual, in order to get super multisolitons of the SUSY BLMP equation (7), we use the ϵ -expansion of g

$$g = 1 + \epsilon g_1 + \epsilon^2 g_2 + \cdots = \sum_{k=0}^{\infty} \epsilon^k g_k$$

and substitute this expansion in the super bilinear form (31). For the super travelling wave or one super soliton solution, we take

$$g_1 = e^{\phi_1}, \quad \phi_1 = \kappa_1 x + \rho_1 y + \omega_1 t + \theta \zeta_1 = \varphi_1 + \theta \zeta_1$$

and $g_i = 0$ for all $i \geq 2$. Introducing g in the super bilinear form (31), we get the dispersive relation $\omega_1 = -\kappa_1^3$. For the two super soliton solution, we take

$$g_1 = e^{\phi_1} + e^{\phi_2}, \quad g_2 = A_{12} \left(1 + 2 \frac{\zeta_1 \zeta_2}{\rho_2 - \rho_1} \right) e^{\varphi_1 + \varphi_2 + \theta(\zeta_1 \alpha_{12} + \zeta_2 \alpha_{21})} \quad (32)$$

and $g_i = 0$ for all $i \geq 3$. In this case, we get the even relations for $i = 1, 2$

$$\omega_i = -\kappa_i^3, \quad A_{12} = \frac{(\kappa_1 - \kappa_2)(\rho_1 - \rho_2)}{(\kappa_1 + \kappa_2)(\rho_1 + \rho_2)}, \quad \alpha_{ij} = \frac{\rho_i + \rho_j}{\rho_i - \rho_j}. \quad (33)$$

The generalization to super multisoliton solutions is direct following the procedure described in [16]. An interesting feature of the above solution is the absence of fermionic constraints.

We refer the readers to [12, 13, 15, 16, 17, 18, 19, 21, 23, 24] for more details on the construction of supersymmetric multisoliton solutions.

4.2. Superpartners

In section 2, we have presented numerous classical solutions of the BLMP equation (2). Here we get for some of them, the superpartner $\xi = \zeta k$, where ζ is an odd constant and $k = k(x, y, t)$ using $\Phi = \xi + \theta u$.

Let us consider solution (22) which takes the form

$$u_{(d_1, d_2)} = -a(1 + \alpha) + \frac{2a(d_1 - d_2)}{d_1 - d_2 + (d_1 + d_2)e^w} - m(y), \quad (34)$$

where $d_1 = c_1 + c_2$ and $d_2 = c_1 - c_2$. In this case, defining $\psi = \xi_w$, we see that $\psi = \psi(w)$ satisfies the time independent Schrödinger equation [4, 8, 9]

$$\psi_{ww} = \left(4 - \frac{3(d_1^2 - d_2^2)}{2(d_1 \cosh(\frac{w}{2}) + d_2 \sinh(\frac{w}{2}))^2} \right) \psi. \quad (35)$$

It can be solved using the change of variable $\mu = \tanh(\frac{w}{2})$. We get the solution $\xi = \zeta k$ where $\zeta^2 = 0$ and

$$\begin{aligned} k_{(d_1, d_2)}(w) = & \beta_1(4d_1 d_2(11d_1^2 + d_2^2) \cosh \frac{w}{2} + 15d_1 d_2(d_1^2 - d_2^2) \cosh \frac{3w}{2} \\ & + 5d_1 d_2(d_2^2 - d_1^2) \cosh \frac{5w}{2} + 4d_2^2(11d_1^2 + d_2^2) \sinh \frac{w}{2} \end{aligned}$$

$$\begin{aligned}
& -5(d_1^2 - d_2^2)(4d_1^2 - d_2^2) \sinh \frac{3w}{2} - (d_1^2 - d_2^2)(4d_1^2 + d_2^2) \sinh \frac{5w}{2} / \\
& (d_1 \cosh \frac{w}{2} + d_2 \sinh \frac{w}{2}) + \beta_2(4d_1(5d_1^2 - d_2^2)(d_1^2 + d_2^2) \cosh \frac{w}{2} + \\
& 5d_1(d_1^2 - d_2^2)(5d_1^2 - 3d_2^2) \cosh \frac{3w}{2} + 5d_1(d_1^4 - d_2^4) \cosh \frac{5w}{2} + \\
& 4d_2(5d_1^2 - d_2^2)(d_1^2 + d_2^2) \sinh \frac{w}{2} + 5d_2(d_2^4 - d_1^4) \sinh \frac{3w}{2} + \\
& d_2(d_1^2 - d_2^2)(9d_1^2 + d_2^2) \sinh \frac{5w}{2}) / (d_1 \cosh \frac{w}{2} + d_2 \sinh \frac{w}{2}) + \beta_3,
\end{aligned}$$

where the β_i 's are constants of integration.

Let us give some relevant special solutions. The case $(d_1, d_2) = (1, 0)$ (corresponding to the Pöschl-Teller potential [26] in the Schrödinger equation), we get the kink-type (soliton) solution

$$u_{(1,0)} = -a \tanh\left(\frac{w}{2}\right) - a\alpha - m(y)$$

and the corresponding superpartner $\xi_{(1,0)} = \zeta k_{(1,0)}$ where

$$\begin{aligned}
k_{(1,0)}(w) &= \beta_3 - 8\beta_1(4 \sinh(w) + \sinh(2w) - 2 \tanh(\frac{w}{2})) \\
&+ 10\beta_2(4 \cosh(w) + \cosh(2w)).
\end{aligned}$$

The case $(d_1, d_2) = (0, 1)$ leads to

$$u_{(0,1)} = -a \coth\left(\frac{w}{2}\right) - a\alpha - m(y)$$

with corresponding superpartner $\xi_{(0,1)} = \zeta k_{(0,1)}$ where

$$k_{(0,1)}(w) = \beta_3 + 2(\beta_1 - \beta_2)(-4 \cosh(w) + \cosh(2w)).$$

The above solutions have never been considered and we want to insist on the novelty which is due to a new approach of solving SUSY models.

5. Bilinear Bäcklund transformations

Bäcklund transformations [3, 4, 8, 9, 11, 20, 27, 28, 29] has been shown to be useful in constructing new solutions from known ones and give nonlinear superposition formulas.

The above sections have exhibited new solutions of the classical and SUSY BLMP equation. Here we propose a nice application of the SUSY Bell polynomials applied to the SUSY case. Indeed, we propose to find bilinear Bäcklund transformations. To achieve this, we define

$$L(\Phi) = D_x \Phi_{yt} + D_x \Phi_{xxxy} - 3D_x D_y (\Phi_{xx} D_y \Phi_x + \Phi_{xx} D_y D_x \Lambda)$$

and we suppose that $\psi = -2 \ln \tau$ and $\eta = -2 \ln \mu$ are such that $L(\psi) = L(\eta) = 0$. In order to use the super binary Bell polynomials, we define new variables

$$v = -\frac{\psi - \eta}{2} = \ln \left(\frac{\tau}{\mu} \right), \quad w = -\frac{\psi + \eta}{2} = \ln(\tau\mu) \quad (36)$$

and, in terms of v and w , we have the following equation

$$\begin{aligned} L(\psi) - L(\eta) = & -2(D_x v_{yt} + D_x v_{xxx y} \\ & + 3D_x D_y (v_{xx} D_y w_x + w_{xx} D_y v_x - v_{xx} D_y D_x \Lambda)) = 0. \end{aligned}$$

In order to get bilinear Bäcklund transformations of the SUSY BLMP equation, we fix the suitable constraint

$$\mathcal{Y}_{(x,0,0),(0,1)}(v, w) - D_y D_x \Lambda = D_y w_x + v_x D_y v - D_y D_x \Lambda = 0.$$

Then one can show, using the above constraint, that

$$L(\psi) - L(\eta) = -2D_x (\mathcal{Y}_{(0,0,t),(0,0)}(v, w) + \mathcal{Y}_{(3x,0,0),(0,0)}(v, w))_y$$

and thus we impose the polynomial constraints

$$\mathcal{Y}_{(x,0,0),(0,1)}(v, w) - D_y D_x \Lambda = 0, \quad \mathcal{Y}_{(0,0,t),(0,0)}(v, w) + \mathcal{Y}_{(3x,0,0),(0,0)}(v, w) = 0,$$

which leads, together with the change of variables (36), to a set of bilinear transformation

$$(\mathcal{S}_y \mathcal{D}_x - D_y D_x \Lambda)(\tau \cdot \mu) = 0, \quad (\mathcal{D}_t + \mathcal{D}_x^3)(\tau \cdot \mu) = 0. \quad (37)$$

The above transformation is similar to the bilinear form of the SUSY mKdV equation proposed in [29] identifying the variable y with the variable x . It can be viewed as a SUSY generalization of a bilinear Miura transformation. Furthermore, taking $\tau(x, y, t; \theta) = f(x, y, t)$ and $\mu(x, y, t; \theta) = g(x, y, t)$ in (37), one gets the set of bilinear transformations obtained in [11] for the classical BLMP equation given by

$$(\mathcal{D}_y \mathcal{D}_x - c'(y))(f \cdot g) = 0, \quad (\mathcal{D}_t + \mathcal{D}_x^3)(f \cdot g) = 0. \quad (38)$$

We thus use this transformation to get a set of bilinear Bäcklund transformations of the SUSY BLMP equation. Indeed, we have the following proposition:

Proposition Suppose that (τ, μ) is a solution of the bilinear equations (37), then (τ', μ') satisfying the following bilinear relations

$$\mathcal{D}_x(\tau \cdot \mu') - \alpha \mathcal{D}_x(\mu \cdot \tau') = \beta \tau \mu' - \alpha \beta \tau' \mu, \quad (39)$$

$$\mathcal{S}_y(\tau \cdot \mu') + \alpha \mathcal{S}_y(\mu \cdot \tau') = \gamma \tau \mu' + \alpha \gamma \tau' \mu, \quad (40)$$

$$(\mathcal{D}_t + \mathcal{D}_x^3 - 3\beta \mathcal{D}_x^2 + 3\beta^2 \mathcal{D}_x)(\tau \cdot \tau') = 0, \quad (41)$$

$$(\mathcal{D}_t + \mathcal{D}_x^3 - 3\beta \mathcal{D}_x^2 + 3\beta^2 \mathcal{D}_x)(\mu \cdot \mu') = 0, \quad (42)$$

solves the bilinear equations (37), where α and β are even constants and γ is an odd constant.

Proof: The proof of this proposition is similar to the one presented in [29]. Indeed, we take

$$\mathcal{P}_1 = 2(\tau' \mu' (\mathcal{S}_y \mathcal{D}_x - D_y D_x \Lambda)(\tau \cdot \mu) - \tau \mu (\mathcal{S}_y \mathcal{D}_x - D_y D_x \Lambda)(\tau' \cdot \mu')),$$

$$\mathcal{P}_2 = \tau' \mu' (\mathcal{D}_t + \mathcal{D}_x^3)(\tau \cdot \mu) + \tau \mu (\mathcal{D}_t + \mathcal{D}_x^3)(\tau' \cdot \mu'),$$

and show that $\mathcal{P}_1 = \mathcal{P}_2 = 0$ using the above bilinear relations. The proof follows noticing that \mathcal{P}_1 is equivalent to

$$\mathcal{P}_1 = 2(\tau' \mu' \mathcal{S}_y \mathcal{D}_x(\tau \cdot \mu) - \tau \mu \mathcal{S}_y \mathcal{D}_x(\tau' \cdot \mu')).$$

The above Bäcklund transformations associated to the SUSY BLMP equation are related to the ones for the SUSY mKdV equation [29] and the $\mathcal{N} = 2$ SUSY KdV equation [20] when one identifies the variable y with the variable x .

6. Conclusion

We have started this paper, with an extensive presentation of new solutions of the BLMP equation. Indeed, we have produced using Hirota and Wronskian methods numerous solutions, *e.g.* rational, soliton, positon, negaton solutions. Most solutions being introduced for the first time.

We also presented a new supersymmetric Boiti-Leon-Manna-Pempenelli equation. We have studied the SUSY BLMP equation from different view points using the beauty of the SUSY Bell polynomials. The novelty of this paper is the new approach consisting of finding the superpartners based on solving the linear equation (35)

Futur work should turn around the application of the super Bell polynomials to $\mathcal{N} = 2$ extensions of nonlinear PDE's *e.g.* the SUSY KdV equation [30].

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